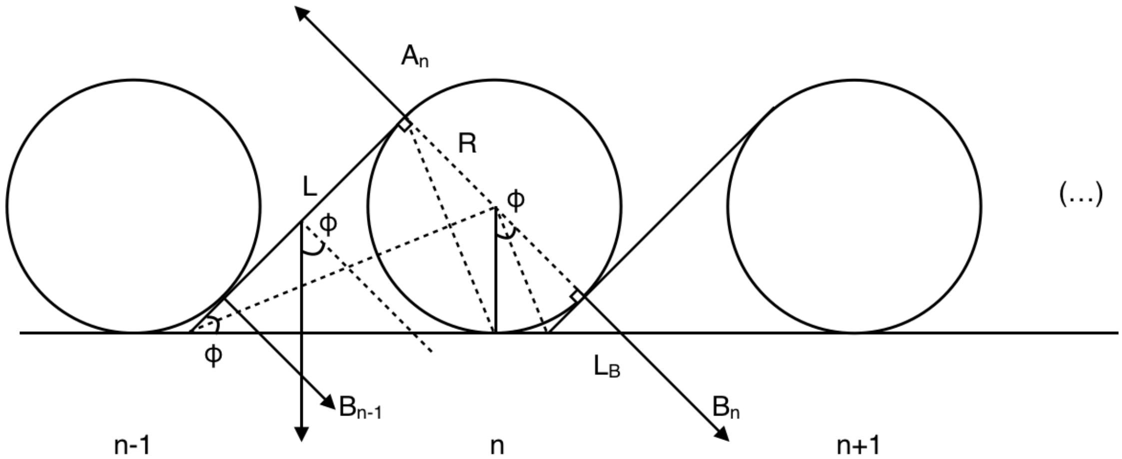
A large number of sticks (with mass density per unit length p and circles (with radius R) lean on each other. Each stick makes an angle φ with the horizontal and is tangent to the next circle at its upper end. The sticks are hinged to the ground, and every other surface is frictionless. In the limit of a very large number of sticks and circles, what is the normal force between a stick and the circle it rests on, very far to the right? Assume that the last circle leans against a wall, to keep it from moving.

The problem asks for the force on a ball arbitrary far to the right. Hence, there is reason to suspect that a general form for the force of a stick leaning on ball can be found in terms of n. Its limit as n tends to infinity can be used to answer the problem, perhaps in the form of a geometric or p- series. We observe an arbitrary set consisting of one stick leaning on one ball, indexed n counting from the left. We need not worry about any net force of the entire system towards the right, as this will be immediately negated by the leftwards normal force at the last ball. Similarly, normal forces by the ground on the stick and ball will be sufficient to negate any net downwards or upwards forces – the solution will be developed mainly through torque analysis.



The stick is acted upon by the forces :

1. pgL, its weight, directed downwards from the midpoint of the stick.
2. B_{n-1}, the normal force by the ball in the preceding set. It is directed perpendicular to the stick. We define this to act on a distance L_B from the hinge.
3. A_n, the normal force exerted by ball in the same set, on the stick. It is directed perpendicular to the stick, and at a distance L from the hinge, at the end of the stick.
4. N, normal force upwards from the ground on the stick.
5. F, sideways force from hinge that keeps stick from sliding rectilinearly towards left.

In a torque analysis, we use contact point C as a pivot and can disregard the 4th and 5th forces, as they act on C. The analysis states :

$$\frac{pgL^2}{2}\cos\varphi + L_B B_{n-1} = A_n L$$

Observing the geometry of the stick, a relation is found for L_B and substituted :

$$\tan\frac{\varphi}{2} = \frac{L_B}{R}$$

$$\frac{pgL^2}{2}\cos\varphi + R\tan\frac{\varphi}{2}B_{n-1} = A_nL$$

In order to relate B_{n-1} and A_n, we observe torques acting on the rim of the ball. The pivot is in this case the geometric center of the circle. A_n is the upwards-pushing torque by stick n on ball n, while B_{n+1} is the upwards-pushing torque by stick $n+1$ on ball n. They are the only two forces constituting torque, and both act at R from the pivot. Hence :

$$A_n = B_{n+1}$$

Considering this in the previous expression :

$$B_{n+1}L = \frac{pgL^2}{2}\cos\varphi + R\tan\frac{\varphi}{2}B_{n-1}$$

$$B_{n+1} = \frac{pgL}{2}\cos\varphi + \frac{R}{L}\tan\frac{\varphi}{2}B_{n-1}$$

Note that no ball learns on the first stick, and hence $B_0 = 0$. Finding successive terms :

$$\begin{aligned}
B_2 &= \frac{pgL}{2}\cos\varphi \\
B_4 &= \frac{pgL}{2}\cos\varphi + \frac{R}{L}\tan\frac{\varphi}{2}B_2 \\
&= \frac{pgL}{2}\cos\varphi + \frac{R}{L}\tan\frac{\varphi}{2}\left[\frac{pgL}{2}\cos\varphi\right] \\
B_5 &= \frac{pgL}{2}\cos\varphi\left(1 + \frac{R\tan(\varphi/2)}{L} + \left[\frac{R\tan(\varphi/2)}{L}\right]\hat{}2\right)
\end{aligned}$$

A general pattern emerges, showing :

$$B_n = \frac{pgL}{2}\cos\varphi\left(1 + \frac{R\tan(\varphi/2)}{L} + \left[\frac{R\tan(\varphi/2)}{L}\right]^2 + \cdots + \left[\frac{R\tan(\varphi/2)}{L}\right]^{n-1}\right)$$

This resembles a geometric series, so long as the ratio term is < 1. The condition for a finite force is hence :

$$\tan(\varphi/2) < \frac{L}{R}$$

The solution for the above geometric series is then :

$$\lim_{n\to\infty} B_n = \frac{pgL}{2}\cos\varphi\left(\frac{L}{L - R\tan(\varphi/2)}\right)$$

The final solution should not contain any L terms. Observing the circle, a trigonometric relation can be found :

$$\mathrm{L} = \frac{R}{\tan(\varphi/2)}$$

Substituting in :

$$\lim_{n\to\infty} B_n = \frac{pg\cos\varphi}{2}\left(\frac{R^2\tan(\varphi/2)}{R - R\tan^2(\varphi/2)}\right)\frac{1}{\tan^2(\varphi/2)}$$

$$= \frac{pg\cos\varphi}{2}\left(\frac{R\cos^2(\varphi/2)}{\cos\varphi}\right)\frac{1}{\tan(\varphi/2)}$$

$$= \frac{pgR}{2}\left(\frac{\cos^3(\varphi/2)}{\sin(\varphi/2)}\right)$$

Which is the answer to be found.