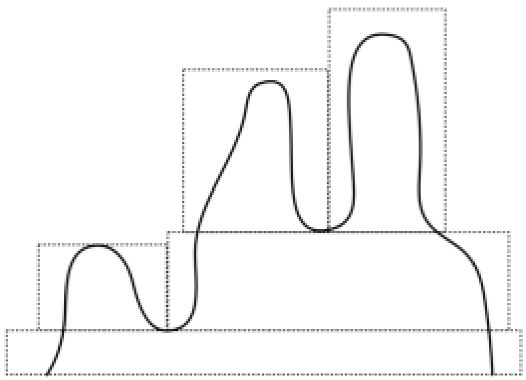
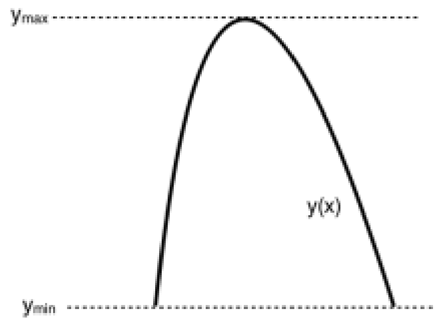
A rigid tube is placed in the vertical plane. It has the shape of an arbitrary curve, but has start and end points of the same height. A chain with uniform mass distribution lies in the tube. Show that the chain lies stationary in the tube, no matter its shape.

By definition, if any section of the chain passes through some height section, there must exist another section of the chain passing through the same height section, but with the opposite gradient. We can pair up height sections suchly. This allows us to compare leftwards force, pulling downwards on the chain when its gradient is positive, with rightwards force, which pulls downwards when its gradient is negative.



Pairing up sections in a possible curve



A simplified curve of interest

We will now investigate one such pair sharing the same $y_{minimum}$ and $y_{maximum}$. Let each y-coordinate is passed through only twice – once on the way up, and once on the way down. For the chain to be stationary, the total leftwards force by the sum mass of the chain length on the maxima's left must be equal to that on the right. We guess that the solution will be of the form :

$$F_{left} = F_{right}$$
$$\int_0^h F_{left} \cdot dy = \int_0^h F_{right} \cdot dy$$

Let mass be distributed along the chain by uniform distribution :

$$\mu = \frac{dm}{ds}$$

The chain has an arc length that is related to change in x- and y- components :

$$ds^2 = dx^2 + dy^2$$
$$\left(\frac{\mathrm{d}s}{\mathrm{d}y}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1$$
$$\frac{\mathrm{d}s}{\mathrm{d}y} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$

Investigating the infinitesimal mass element dm, we define it to subtend angle θ with the horizontal. We find it experiences the following forces :

1. Force parallel to plane, pointing away from maxima : $\mathrm{d}mg\sin\theta = \mu\mathrm{d}sg\sin\theta$
2. Force perpendicular, pointing into plane $\mathrm{d}mg\cos\theta = \mu\mathrm{d}sg\cos\theta$. This is stabilized by normal force provided by rigid tube, and does not have to be considered as the tube is frictionless.

The total active force pulling away from the maxima is given by component (1) only. We present it in terms of dy as we foresee an easy integration from two shared $y_{minimum}$ and $y_{maximum}$ heights.

$$\mu \mathrm{d}s g \sin\theta = \mu g \sin\theta \, \mathrm{d}y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$

Total force from $y_{minimum}$ to $y_{maximum}$ is the integration :

$$F_{tot} = \int_{y_{min}}^{y_{max}} \mu g \sin\theta \, \mathrm{d}y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$

We know that θ is related to gradient, dy/dx by the following :

$$\tan\theta = \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{1}{\sin^2\theta} = \frac{1}{\tan^2\theta} + 1$$
$$= \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1$$

We can substitute $\sin\theta$ in terms of dy/dx:

$$\sin\theta = \frac{1}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}}$$

And obtain the expression for total force :

$$F_{tot} = \int_{y_{min}}^{y_{max}} \mu g \frac{1}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}} \mathrm{d}y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$
$$= \int_{y_{min}}^{y_{max}} \mu g \, \mathrm{d}y$$
$$= \mu g \, \Delta y$$

Clearly, the gravitational force parallel to the plane depends only on the change in height spanned by the chain section. For opposite-gradient pair with at the same $y_{minimum}$ and $y_{maximum}$, force leftwards by the positive gradient section, and force rightwards by the negative gradient section, are equal. As the chain as defined is composed of such pairs, its total force rightwards is equal to its total force leftwards, and it remains stationary.